Lecture 9 02/14/2018

Review of Magnetostatics

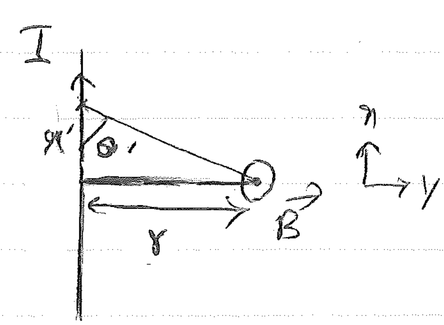
Moving charges produce magnetic field. The Biot-Savart law describes the magnetic field due to a moving point charge q:

$$\vec{B} = \frac{\mu_0}{4\pi} q \frac{\vec{v} \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \qquad \left(\mu_0 = \frac{1}{\varepsilon_0 c^2}\right)$$

This relation is approximate and is valid at non-relativistic speeds. In general, for a current distribution $\vec{J}(\vec{x}')$, we have:

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \, d\tau'$$

As an example, let us consider an infinite straight wire carrying a current I. In this case:



$$\vec{J}(\vec{x}') \, d\tau' = I \, dx' \, \hat{x}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{dx' \sin\theta'}{(x'^2 + r^2)} \, \hat{z} =$$

$$\frac{\mu_0 I}{2\pi r} \int_0^{\frac{\pi}{2}} \cos\theta' \, d\theta' \hat{z} = \frac{\mu_0 I}{2\pi r} \hat{z}$$

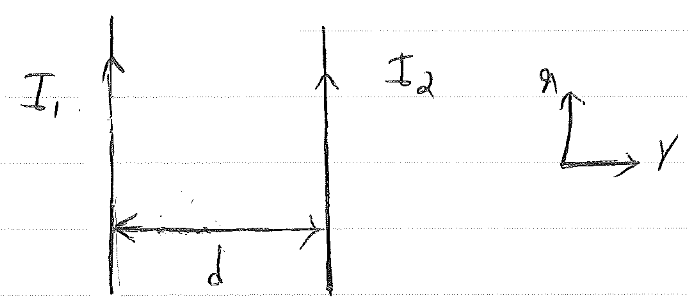
Magnetic fields exert force on moving charges. For a point charge q moving at velocity $\vec{v}$, the Lorentz force is:

$$\vec{F} = q\vec{v} \times \vec{B}$$

For a general current distribution, the total force is:

$$\vec{F} = \int \vec{J}(\vec{x}') \times \vec{B}(\vec{x}') \, d\tau'$$

For example, consider two parallel infinite wires carrying currents I_1 and I_2, respectively, shown here.



The force per unit length of wire 2 is given by:

$$\vec{F}_2 = -I_2 \int_0^1 dl_2 \, B_1(\vec{x}_2) \hat{y} = \frac{-\mu_0 I_1 I_2}{2\pi d} \hat{y}$$

We note that $B_2(\vec{x}_2) = 0$ anywhere on wire 2. Similarly, it can be shown that:

$$\vec{F}_1 = I_1 \int dl_1 \, B_2(\vec{x}_1) \hat{y} = \frac{\mu_0 I_1 I_2}{2\pi d} \hat{y}$$

As mentioned, magnetic field due to a current distribution $\vec{J}$ is:

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \, d\tau'$$

However:

$$\frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} = -\vec{\nabla}\left(\frac{1}{|\vec{x} - \vec{x}'|}\right) \Rightarrow B(\vec{x}) = \frac{-\mu_0}{4\pi} \int \vec{J}(\vec{x}') \times \vec{\nabla}\left(\frac{1}{|\vec{x} - \vec{x}'|}\right) d\tau'$$

$$= \frac{\mu_0}{4\pi} \vec{\nabla} \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \, d\tau'$$

Here, we have used:

$$\vec{\nabla} \times \left(\vec{J}(\vec{x}') \frac{1}{|\vec{x} - \vec{x}'|}\right) = -\vec{J}(\vec{x}') \times \vec{\nabla}\left(\frac{1}{|\vec{x} - \vec{x}'|}\right) + \frac{1}{|\vec{x} - \vec{x}'|} \cancelto{0}{\vec{\nabla} \times \vec{J}(\vec{x}')}$$

0 (since $\vec{\nabla}$ does not operate on $\vec{x}'$)

Therefore, we can write:

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad , \quad \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \, d\tau'$$

An important Consequence of $\vec{B} = \vec{\nabla} \times \vec{A}$ is that:

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \text{no magnetic monopoles}}$$

The quantity $\vec{A}(\vec{x})$ is called the <u>vector potential</u>. However, we note that $\vec{A}(\vec{x})$ is not unique. In fact, if $\vec{B} = \vec{\nabla} \times \vec{A}$, then

$\vec{A}' = \vec{A} + \vec{\nabla}\chi$ (where χ is an arbitrary scalar function) will give exactly the same $\vec{B}$. This is called a gauge transformation and implies that $\vec{A}$ is not a physically observable quantity.

There are a couple of important properties for the current density $\vec{J}(\vec{x})$ in magnetostatics. First, conservation of the electric charge implies that:

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Since there is no explicit time dependence in the case of magnetostatics, $\frac{\partial \rho}{\partial t} = 0$, and hence:

$$\boxed{\vec{\nabla}\cdot\vec{J} = 0}$$

Second, consider a localized charge distribution for which $J(\vec{x})$ drops quickly as $r = |\vec{x}|$ grows. In this case:

$$\int_V \partial_i (\underbrace{x_j J_i}_{\text{Einstein summation convention implied}}) \, d\tau = \int_V \vec{\nabla}\cdot\vec{K} \, d\tau = \oint_S \vec{K}\cdot\hat{n}\, da = \oint_S x_j J_i n_i \, da$$

$K_i \equiv x_j J_i$

Taking the boundary S to infinity, since $J(\vec{x})$ is localized, results in the following:

$$\int_V \partial_i (x_j J_i)\, d\tau = 0$$

However:

$$\int_V \partial_i (x_j J_i)\, d\tau = \int_V (\partial_i x_j) J_i\, d\tau + \int_V x_j \overset{\vec{\nabla}\cdot\vec{J}=0}{\overbrace{(\partial_i J_i)}}\, d\tau \Rightarrow$$

$$\int_V (\partial_i x_j) J_i\, d\tau = \int_V \delta_{ij} J_i\, d\tau = \int_V J_j\, d\tau = 0 \Rightarrow \boxed{\int_V \vec{J}\, d\tau = 0}$$

Writing $\vec{B} = \vec{\nabla} \times \vec{A}$, we find:

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2 \vec{A}$$

But:

$$\vec{\nabla}\cdot\vec{A} = \frac{\mu_0}{4\pi} \vec{\nabla}\cdot\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d\tau' = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\cdot\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d\tau'$$

$$= -\frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d\tau'$$

However:

$$\vec{J}(\vec{x}') \cdot \vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \vec{\nabla}' \cdot \left(\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right) - \cancel{\frac{\vec{\nabla}' \cdot \vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}}^{\,0}$$

Therefore:

$$\vec{\nabla} \cdot \vec{A} = -\frac{\mu_0}{4\pi} \int \vec{\nabla}' \cdot \left(\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right) d\tau' = 0$$

for a localized current distribution (using the divergence theorem)

This results in:

$$\vec{\nabla} \times \vec{B} = -\nabla^2 \vec{A} = -\frac{\mu_0}{4\pi} \nabla^2 \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d\tau' = -\frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d\tau'$$

$$= \mu_0 \int \vec{J}(\vec{x}') \, \delta^{(3)}(\vec{x}-\vec{x}') \, d\tau'$$

This gives rise to the Ampere's law for magnetostatics:

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$$

Magnetic Fields of a Localized Current Distribution

The vector potential due to a localized steady current distribution $J(\vec{x})$ is:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d\tau' = \frac{\mu_0}{4\pi} \int \left[\frac{1}{r} + \frac{\vec{x}\cdot\vec{x}'}{r^3} + O\left(\frac{r'^2}{r^3}\right)\right] J(\vec{x}') \, d\tau'$$

However:

$$\frac{\mu_0}{4\pi}\int \frac{1}{r}\vec{J}(\vec{x}')\,d\tau' = \frac{\mu_0}{4\pi r}\underbrace{\int \vec{J}(\vec{x}')\,d\tau'}_{\substack{\text{vanishes for a} \\ \text{localized distribution}}} = 0$$

This is just a manifestation that there are no magnetic monopoles.

Therefore, the first non-vanishing term is:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{\vec{x}}{r^3}\cdot\int \vec{x}'\,\vec{J}(\vec{x}')\,d\tau' + \dots$$

We can write:

$$\vec{x}\cdot\int \vec{x}'\,\vec{J}(\vec{x}')\,d\tau' = \hat{e}_i\, x_j \int x'_j\, J_i(\vec{x}')\,d\tau'$$ ($\hat{e}_i$: unit vector along the i-th direction)

$$x'_j J'_i(\vec{x}') = \frac{1}{2}\underbrace{(x'_j J_i + x'_i J_j)}_{\substack{\text{symmetric} \\ \text{under } i\leftrightarrow j}} + \frac{1}{2}\underbrace{(x'_j J_i - x'_i J_j)}_{\substack{\text{anti-symmetric} \\ \text{under } i\leftrightarrow j}}$$

Now:

$$\int (x'_j J_i + x'_i J_j)\,d\tau' = \int \left[x'_j \frac{\partial}{\partial x'_k}(x'_i J_k) + x'_i J_j\right] d\tau'$$

Here, we have used:

$$\frac{\partial}{\partial x'_k}(x'_i J_k) = \delta_{ik} J_k + x'_i \underbrace{\partial'_k J_k}_{\vec{\nabla}\cdot\vec{J}=0} = J_i$$

Note that:

$$x'_j \frac{\partial}{\partial x'_k}(x'_i J_k) = \frac{\partial}{\partial x'_k}(x'_j x'_i J_k) - \left(\frac{\partial x'_j}{\partial x'_k}\right) x'_i J_k$$

Thus:

$$\int \left[x'_j \frac{\partial}{\partial x'_k}(x'_i J_k) + x'_i J_j \right] d\tau' = \underbrace{\oint_S x'_j x'_i \vec{J} \cdot \hat{n}' \, da'}_{\text{vanishes for a localized distribution}} - \int x'_i \cancel{J_j} \, d\tau'$$

$$+ \int x'_i \cancel{J_j} \, d\tau'$$

As a result:

$$\vec{x} \cdot \int \vec{x}' \vec{J}(\vec{x}') \, d\tau' = \frac{1}{2} \hat{e}_i x_j \int \left[x'_j J_i(\vec{x}') - x'_i J_j(\vec{x}') \right] d\tau'$$

Hence:

$$\vec{A}(\vec{x}) = \frac{-\mu_0}{4\pi r^3} \vec{x} \times \left(\frac{1}{2} \int \vec{x}' \times \vec{J}(\vec{x}') \, d\tau' \right) + \ldots \Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi r^3} \vec{m} \times \vec{x} + \ldots$$

Here, $\vec{m}$ is the <u>magnetic dipole moment</u> of the distribution:

$$\vec{m} \equiv \frac{1}{2} \int \vec{x}' \times \vec{J}(\vec{x}') \, d\tau'$$

The magnetic moment density is defined as:

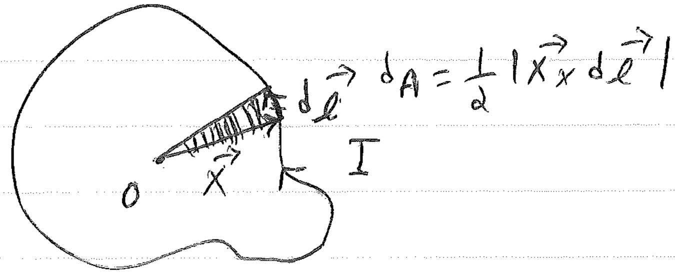
$$\vec{\mu}(\vec{x}) = \frac{1}{2} \vec{x} \times \vec{J}(\vec{x})$$

For a current loop carrying a current I, we have:

$$\vec{m} = \frac{I}{2} \oint_C \vec{x} \times d\vec{\ell}$$

For a planar loop, $\vec{x}\times d\vec{\ell}$ is in the direction of the normal to the loop, $\hat{n}$, and its magnitude is $2\,dA$:

$$\vec{m} = I\int dA\ \hat{n} = IA\hat{n}$$



For a point magnetic dipole at the origin, the expression $\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{r^3}$ is exact. Then:

$$\vec{B}(\vec{x}) = \vec{\nabla}\times\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\nabla\times\left(\frac{\vec{m}\times\vec{x}}{r^3}\right) = -\frac{\mu_0}{4\pi}\vec{\nabla}\times\left(\vec{m}\times\vec{\nabla}\left(\frac{1}{r}\right)\right) =$$

$$-\frac{\mu_0}{4\pi}\left[\left(\vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{r}\right)\right)\vec{m} - (\vec{m}\cdot\vec{\nabla})\vec{\nabla}\left(\frac{1}{r}\right)\right]$$

We note that $\nabla^2\left(\frac{1}{r}\right) = 4\pi\delta^{(3)}(\vec{x})$. Therefore, at any point other than the origin, we have:

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}(\vec{m}\cdot\vec{\nabla})\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\mu_0}{4\pi}(\vec{m}\cdot\vec{\nabla})\left(\frac{\vec{x}}{r^3}\right) = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\vec{x})\vec{x}}{r^5}\right.$$

$$\left.-\frac{\vec{m}}{r^3}\right] = \boxed{\frac{\mu_0}{4\pi}\left[\frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{r^3}\right]} \qquad \left(\hat{n}\equiv\frac{\vec{x}}{r}\right)$$